Third post-Newtonian accurate generalized quasi-Keplerian parametrization for compact binaries in eccentric orbits

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We present Keplerian-type parametrization for the solution of third post-Newtonian (3PN) accurate equations of motion for two nonspinning compact objects moving in an eccentric orbit. The orbital elements of the parametrization are explicitly given in terms of the 3PN accurate conserved orbital energy and angular momentum in both Arnowitt-Deser-Misner–type and harmonic coordinates. Our representation will be required to construct post-Newtonian accurate ''ready to use'' search templates for the detection of gravitational waves from compact binaries in inspiralling eccentric orbits. Because of the presence of certain 3PN accurate gauge invariant orbital elements, the parametrization should be useful to analyze the compatibility of general relativistic numerical simulations involving compact binaries with the corresponding post-Newtonian descriptions. If required, the present parametrization will also be needed to compute post-Newtonian corrections to the currently employed ''timing formula'' for the radio observations of relativistic binary pulsars.

I. INTRODUCTION

An accurate analytic description for the orbital dynamics of compact binaries of comparable masses is highly desirable to probe astrophysical scenarios involving strong gravitational fields. These include inspiralling of compact binaries, relevant as sources of gravitational radiation for the operational and proposed interferometric detectors and relativistic binary pulsars.

The method of matched filtering, ideally suited to detect gravitational radiation emitted by inspiralling compact binaries, requires ''ready to use'' search templates [1]. These templates usually consist of gravitational wave polarizations, $h_+(t)$ and $h_\times(t)$, temporally evolving due to the orbital dynamics of the binary. In the inspiral regime, the orbital dynamics is well described by the post-Newtonian (PN) approximation to general relativity, which allows one to express the equations of motion for a compact binary as corrections to Newtonian equations of motion in powers of $(v/c)^2 \sim GM/(c^2 R)$, where v, M, and R are the velocity, total mass, and relative separation of the binary. Currently, widely employed search templates are for compact objects moving in quasicircular orbits, where $h_+(t)$ and $h_\times(t)$ are known, both in amplitude and phase evolution, to second post-Newtonian (2PN) order [2]. However, it is highly desirable to know *a priori* the temporal phase evolution of $h_+(t)$ and $h_\times(t)$ at least up to third post-Newtonian (3PN) order beyond the (Newtonian) quadrupole contribution [3], which in turn requires the knowledge of the conservative binary dynamics to 3PN order, i.e., $(v/c)^6$ corrections to Newtonian equations of motion. Very recently, after tackling for years many conceptual and computational issues, the phase evolution of gravitational wave polarizations to 3PN order was achieved [see [4] and references therein].

Also recently, a formalism to compute ready to use search templates for compact binaries in inspiralling eccentric orbits was developed and explicitly implemented at 2.5PN order [5]. Its natural extension to higher PN orders, which should be relevant for detecting compact binaries in eccentric orbits, requires a parametric solution to the 3PN accurate equations of motion.

To describe the late stages of binary inspiral and the subsequent merger, the emphasis is currently being placed on numerical relativity, which attempts to solve the associated set of full Einstein equations using supercomputers [6]. However, these general relativistic simulations usually do not incorporate inputs from PN dynamics and are incapable of describing few binary orbits even in the inspiral regime. Recently, a ''post-Newtonian diagnostic tool'' was introduced with the aim of extracting some physical information from the existing general relativistic simulations [7,8]. This ''diagnostic tool'' requires a solution of 3PN accurate equations of motion for compact binaries moving in noncircular orbits. The tool further introduced certain definitions for the eccentricity and semilatus rectum parameters in terms of the orbital angular velocity at the turning points of the orbit, making many of its estimates gauge dependent. It is desirable to characterize a noncircular orbit in post-Newtonian relativity in terms of gauge invariant quantities and the parametrization we present here should be useful in that aspect.

Finally, we note that the high precision radio-wave observations of binary pulsars employ an accurate relativistic ''timing formula'' [9,10] which requires an analytic solution to the relativistic equations of motion for a compact binary moving in an elliptical

orbit [11]. This timing formula is instrumental to test both the predictions of general relativity and the viability of alternate theories of gravity in strong field situations [12]. It should be noted that the long term measurements of the recently discovered relativistic double pulsar system J0737-3039 will require the inclusion of higher order PN effects in the timing formula [13].

In this paper, we derive an analytic parametric solution to 3PN accurate conservative equations of motion for compact binaries moving in eccentric orbits. The orbital representation is given both in Arnowitt-Deser-Misner(ADM)−type and harmonic coordinates. We employ similar techniques which allowed Damour and Deruelle to obtain a remarkably simple parametrization for the solution of 1PN accurate equations of motion for compact binaries in eccentric orbits [11]. Further, our 3PN accurate orbital representation is structurally quite close to the generalized quasi-Keplerian representation obtained by Damour, Schäfer, and Wex for the solution of 2PN accurate orbital motion of compact binaries in eccentric orbits in ADM gauge [14,15]. We explicitly demonstrate that, to these high PN orders, a slowly precessing eccentric orbit can be characterized by certain gauge invariant quantities. This feature of the parametrization should make it very attractive to describe eccentric orbits in post-Newtonian relativity. We note that, apart from some misprints, an incomplete representation in ADM-type coordinates was obtained earlier [16].

We have the following plan for the paper. In Sec. II, we first review Keplerian parametrization associated with the Newtonian accurate orbital motion. This will be followed by brief descriptions about the quasi-Keplerian and the generalized quasi-Keplerian parametrizations associated, respectively, with the 1PN and 2PN accurate orbital motion. Section III deals with the derivation of 3PN accurate orbital representation for compact binaries moving in eccentric orbits in ADM-type coordinates. In Sec. IV, a similar parametrization, valid in harmonic gauge, is derived. Finally, in Sec. V, we summarize our results and discuss the merits and applications of the parametrization.

II. THE KEPLERIAN PARAMETRIZATION AND ITS PN EXTENSIONS

The Keplerian parametrization for Newtonian accurate orbital motion of a binary in eccentric orbit is heavily employed in celestial mechanics [17]. In polar coordinates and in the center-of-mass reference frame, the eccentric motion is parametrized in the following way:

$$R = a(1 - e \cos u), \tag{1a}$$

$$\phi - \phi_0 = v \equiv 2 \arctan\left[\left(\frac{1 + e}{1 - e}\right)^{1/2} \tan\frac{u}{2}\right], \tag{1b}$$

where R and ϕ define the components of the relative separation vector $\mathbf{R} = R(\cos\phi, \sin\phi, 0)$. The semimajor axis and the eccentricity of the orbit are denoted by a and e, respectively; both are expressible in terms of the Newtonian conserved (orbital) energy and angular momentum. The auxiliary angles u and v are called eccentric and true anomaly, respectively. The geometrical interpretation of these anomalies is presented in Fig. 1. The explicit time dependence is provided by the Kepler equation, which reads

$$l \equiv n(t - t_0) = u - e \sin u, \tag{2}$$

where l is the mean anomaly and n is referred to as the mean motion and is given by $n = \frac{2\pi}{P}$, P being the orbital period. The quantities t_0 and ϕ_0 are some initial time and orbital phase. The explicit functional dependence of the orbital elements a, e, and n is given by
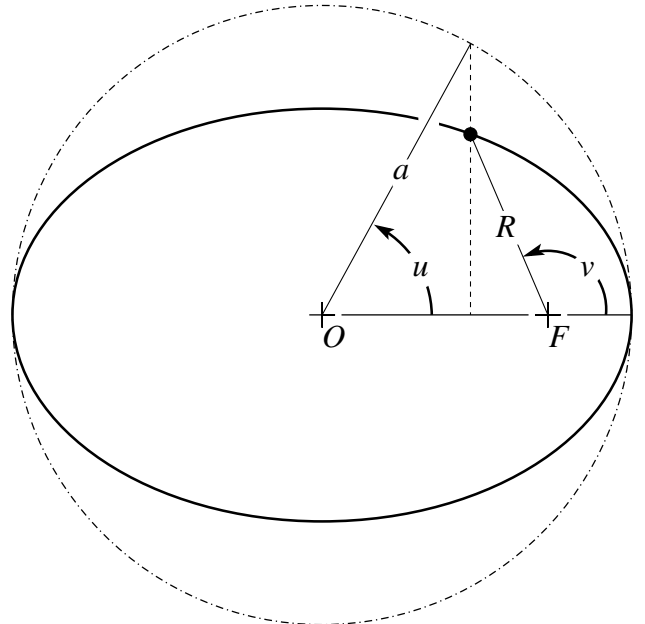

FIG. 1. The geometrical interpretation of the eccentric and true anomalies, u and v, appearing in the Keplerian parametrization. The auxiliary circle of radius a circumscribes the orbital ellipse with semimajor axis a. The points O and F stand for the origin and the focus of the ellipse.

$$a = \frac{GM}{(-2E)}, \tag{3a}$$

$$e^2 = 1 + 2Eh^2, \tag{3b}$$

$$n = \frac{(-2E)^{3/2}}{GM}, \tag{3c}$$

where E is the Newtonian orbital energy per unit reduced mass $\mu = m_1 m_2 / M$, m_1 and m_2 being the individual masses of the binary and $M = m_1 + m_2$. The reduced angular momentum h is given by $h = \frac{J}{GM}$, where J is the reduced Newtonian orbital angular momentum.

For 1PN accurate equations of motion, in harmonic coordinates, Damour and Deruelle found the following "Keplerian-like" parametrization [11], which reads

$$R = a_r(1 - e_r \cos u), \tag{4a}$$

$$l \equiv n(t - t_0) = u - e_t \sin u, \tag{4b}$$

$$\frac{2\pi}{\Phi}(\phi - \phi_0) = v \equiv 2 \arctan\left[\left(\frac{1 + e_\phi}{1 - e_\phi}\right)^{1/2} \tan\frac{u}{2}\right]. \tag{4c}$$

Note that the three eccentricities e_r, e_t, and e_ϕ, which differ from each other by PN corrections in terms of E and the finite mass ratio $\eta = \mu/M$, were introduced such that the parametrization looks "Keplerian" even at 1PN order. The factor $\frac{2\pi}{\Phi}$ gives the angle of advance of the periastron per orbital revolution. Because of these features, in the literature, the above representation is usually referred to as the "quasi-Keplerian" parametrization for 1PN accurate orbital motion. The parameters a_r, e_r, e_ϕ, n, and e_t are some 1PN "semimajor" axis, "radial eccentricity," "angular eccentricity," "mean motion," and "time eccentricity," respectively. These orbital elements depend on the 1PN accurate conserved energy and angular momentum; their explicit dependence was derived in [11].

The above orbital parametrization was extended, in ADM coordinates, to 2PN order by Damour, Schäfer, and Wex [14,15]. The 2PN accurate orbital parametrization has the following form:

$$R = a_r(1 - e_r \cos u), \tag{5a}$$

$$l \equiv n(t - t_0) = u - e_t \sin u + \left(\frac{g_{4t}}{c^4}\right)(v - u)$$
$$+ \left(\frac{f_{4t}}{c^4}\right)\sin v, \tag{5b}$$

$$\frac{2\pi}{\Phi}(\phi - \phi_0) = v + \left(\frac{f_{4\phi}}{c^4}\right)\sin 2v + \left(\frac{g_{4\phi}}{c^4}\right)\sin 3v, \tag{5c}$$

where $v = 2 \arctan\{[(1 + e_\phi)/(1 - e_\phi)]^{1/2} \tan\frac{u}{2}\}$.

The presence of the orbital functions g_{4t}, f_{4t}, $g_{4\phi}$, and $f_{4\phi}$ indicates that the structure of the solution is more general than the quasi-Keplerian one, prompting one to coin the above parametrization as the "generalized quasi-Keplerian" parametrization for 2PN accurate orbital motion for compact binaries in eccentric orbits. The explicit expressions for the orbital elements $a_r, e_r, n, e_t, \Phi, e_\phi$ along with $g_{4t}, f_{4t}, g_{4\phi}$, and $f_{4\phi}$ in terms of the conserved 2PN accurate energy and angular momentum as well as the finite mass ratio were derived, in ADM coordinates, in [15].

In the next two sections, we will derive a similar orbital parametrization for both ADM-type and harmonic gauges, which will *analytically* describe the solution to the 3PN accurate equations of motion for a compact binary moving in an "eccentric" orbit.

III. THE 3PN ACCURATE GENERALIZED QUASI-KEPLERIAN REPRESENTATION IN ADM-TYPE COORDINATES

In the Hamiltonian formulation of general relativity, advocated by Arnowitt, Deser, and Misner [18], an ordinary Hamiltonian for a compact binary is achievable only up to 2PN order [19,20].

At the third post-Newtonian order, a higher order Hamiltonian which depends on particle positions, conjugate momenta, and their derivatives was derived by Jaranowski and Schäfer [21]. Later, using a higher order contact transformation, the above Hamiltonian was transformed into an ordinary Hamiltonian by Damour, Jaranowski, and Schäfer [22]. We refer to this ordinary Hamiltonian as the 3PN Hamiltonian in ADM-type coordinates. The two unknown coefficients, which initially appeared in the above Hamiltonian, were later fixed [23]. We display below the fully determined reduced ordinary 3PN Hamiltonian, in ADM-type coordinates and in the center-of-mass frame, compiled using [23,24] as

$$\mathcal{H}(\mathbf{r}, \hat{\mathbf{p}}) = \mathcal{H}_0(\mathbf{r}, \hat{\mathbf{p}}) + \frac{1}{c^2}\mathcal{H}_1(\mathbf{r}, \hat{\mathbf{p}}) + \frac{1}{c^4}\mathcal{H}_2(\mathbf{r}, \hat{\mathbf{p}})$$
$$+ \frac{1}{c^6}\mathcal{H}_3(\mathbf{r}, \hat{\mathbf{p}}), \tag{6}$$

where the Newtonian and post-Newtonian contributions are given by

$$\mathcal{H}_0(\mathbf{r}, \hat{\mathbf{p}}) = \frac{\hat{\mathbf{p}}^2}{2} - \frac{1}{r}, \tag{7a}$$

$$\mathcal{H}_1(\mathbf{r}, \hat{\mathbf{p}}) = \frac{1}{8}(3\eta - 1)(\hat{\mathbf{p}}^2)^2 - \frac{1}{2}[(3 + \eta)\hat{\mathbf{p}}^2 + \eta(\mathbf{n} \cdot \hat{\mathbf{p}})^2]\frac{1}{r} + \frac{1}{2r^2}, \tag{7b}$$

$$\mathcal{H}_2(\mathbf{r}, \hat{\mathbf{p}}) = \frac{1}{16}(1 - 5\eta + 5\eta^2)(\hat{\mathbf{p}}^2)^3 + \frac{1}{8}[(5 - 20\eta - 3\eta^2)(\hat{\mathbf{p}}^2)^2 - 2\eta^2(\mathbf{n} \cdot \hat{\mathbf{p}})^2\hat{\mathbf{p}}^2 - 3\eta^2(\mathbf{n} \cdot \hat{\mathbf{p}})^4]\frac{1}{r}$$
$$+ \frac{1}{2}[(5 + 8\eta)\hat{\mathbf{p}}^2 + 3\eta(\mathbf{n} \cdot \hat{\mathbf{p}})^2]\frac{1}{r^2} - \frac{1}{4}(1 + 3\eta)\frac{1}{r^3}, \tag{7c}$$

$$\mathcal{H}_3(\mathbf{r}, \hat{\mathbf{p}}) = \frac{1}{128}(-5 + 35\eta - 70\eta^2 + 35\eta^3)(\hat{\mathbf{p}}^2)^4 + \frac{1}{16}[(-7 + 42\eta - 53\eta^2 - 5\eta^3)(\hat{\mathbf{p}}^2)^3$$
$$+ (2 - 3\eta)\eta^2(\mathbf{n} \cdot \hat{\mathbf{p}})^2(\hat{\mathbf{p}}^2)^2 + 3(1 - \eta)\eta^2(\mathbf{n} \cdot \hat{\mathbf{p}})^4\hat{\mathbf{p}}^2 - 5\eta^3(\mathbf{n} \cdot \hat{\mathbf{p}})^6]\frac{1}{r}$$
$$+ \left[\frac{1}{16}(-27 + 136\eta + 109\eta^2)(\hat{\mathbf{p}}^2)^2 + \frac{1}{16}(17 + 30\eta)\eta(\mathbf{n} \cdot \hat{\mathbf{p}})^2\hat{\mathbf{p}}^2 + \frac{1}{12}(5 + 43\eta)\eta(\mathbf{n} \cdot \hat{\mathbf{p}})^4\right]\frac{1}{r^2}$$
$$+ \left\{\frac{1}{192}[-600 + (3\pi^2 - 1340)\eta - 552\eta^2]\hat{\mathbf{p}}^2 - \frac{1}{64}(340 + 3\pi^2 + 112\eta)\eta(\mathbf{n} \cdot \hat{\mathbf{p}})^2\right\}\frac{1}{r^3}$$
$$+ \frac{1}{96}[12 + (872 - 63\pi^2)\eta]\frac{1}{r^4}, \tag{7d}$$

where $\mathbf{r} = \mathbf{R}/(GM)$, $r = |\mathbf{r}|$, $\mathbf{n} = \mathbf{r}/r$, and $\hat{\mathbf{p}} = \mathbf{P}/\mu$; $\mathbf{R}$ and $\mathbf{P}$ are the relative separation vector and its conjugate momentum vector.

The invariance of $\mathcal{H}$ under time translation and spatial rotations leads to the following conserved quantities: The 3PN reduced energy $E = \mathcal{H}$ and the reduced angular momentum $\hat{\mathbf{J}} = \mathbf{r} \times \hat{\mathbf{p}}$ of the binary in the center-of-mass frame. The conservation of $\hat{\mathbf{J}}$ particularly implies that the motion is restricted to a plane and we may introduce polar coordinates such that $\mathbf{r} = r(\cos\phi, \sin\phi)$. The Hamiltonian equations of motion, which govern the relative motion, read

$$\dot{r} = \mathbf{n} \cdot \frac{\partial \mathcal{H}}{\partial \hat{\mathbf{p}}}, \tag{8a}$$

$$r^2\dot{\phi} = \left|\mathbf{r} \times \frac{\partial \mathcal{H}}{\partial \hat{\mathbf{p}}}\right|, \tag{8b}$$

where $\dot{r} = dr/dt$, $\dot{\phi} = d\phi/dt$, and t denotes the coordinate time scaled by GM. We introduce $s = 1/r$ so that $\dot{r}^2 = (ds/dt)^2/s^4 = \dot{s}^2/s^4$ and obtain, using Eqs. (8a) and (8b), 3PN accurate expressions for $\dot{r}^2$ and $d\phi/ds = \dot{\phi}/\dot{s}$, in terms of E, $h = |\hat{\mathbf{J}}|$, η, and s, which are displayed in Appendix A. We note that the 3PN accurate expressions for $\dot{r}^2$ and $\dot{\phi}$ are seventh degree polynomials in s.

To obtain the 3PN accurate orbital parametrization, we proceed as follows. First, we concentrate on the radial motion and compute the two nonzero positive roots, having finite limits as $\frac{1}{c} \to 0$, of the 3PN accurate expression for $\dot{r}^2$. These two 3PN accurate roots, labeled s_- and s_+, correspond to the turning points of the radial motion and, hence, to the periastron and the apastron of the post-Newtonian eccentric orbit. With the help of these two roots, we factorize the 3PN accurate expression for $\dot{r}^2$

and obtain the following integral connecting t and s:

$$t - t_0 = \int_s^{s_-} \frac{A_0 + A_1\bar{s} + A_2\bar{s}^2 + A_3\bar{s}^3 + A_4\bar{s}^4 + A_5\bar{s}^5}{\sqrt{(s_- - \bar{s})(\bar{s} - s_+)\bar{s}^2}}d\bar{s}. \tag{9}$$

The coefficients $A_i : i = 1$–5 are some PN accurate functions of E, h, and η which follow directly from the expression for $\dot{r}^2$. The radial motion is uniquely parametrized by using the ansatz,

$$r = a_r(1 - e_r\cos u), \tag{10}$$

where a_r and e_r are some 3PN accurate semimajor axis and radial eccentricity which may be expressed in terms of s_- and s_+ as

$$a_r = \frac{1}{2}\frac{s_- + s_+}{s_- s_+}, \qquad e_r = \frac{s_- - s_+}{s_- + s_+}. \tag{11}$$

With the aid of above relations and 3PN accurate s_- and s_+, we compute the 3PN accurate expressions for a_r and e_r^2 in terms of E, h, and η. We now move on to obtain the 3PN extension to the "Kepler equation." For this purpose, we first compute the 3PN accurate expression for the radial period P associated with the radial motion. The expression for P directly follows from Eq. (9) and reads

$$P = 2\int_{s_+}^{s_-} \frac{A_0 + A_1\bar{s} + A_2\bar{s}^2 + A_3\bar{s}^3 + A_4\bar{s}^4 + A_5\bar{s}^5}{\sqrt{(s_- - \bar{s})(\bar{s} - s_+)\bar{s}^2}}d\bar{s}. \tag{12}$$

Using Eq. (9), we express the mean anomaly $l \equiv n(t - t_0) = \frac{2\pi}{P}(t - t_0)$ in terms of the eccentric anomaly u. We introduce an auxiliary variable $\tilde{v} = 2\arctan\{[(1 + e_r)/(1 - e_r)]^{1/2}\tan\frac{u}{2}\}$ and with the aid of certain trigonometric relations involving $\tilde{v}$, displayed in Appendix B,

we obtain the following temporary parametrization for the mean anomaly l which has the following general structure:

$$l \equiv n(t - t_0)$$
$$= u + \kappa_0 \sin u + \frac{\kappa_1}{c^2}(\tilde{v} - u) + \frac{\kappa_2}{c^4} \sin \tilde{v}$$
$$+ \frac{\kappa_3}{c^6} \sin 2\tilde{v} + \frac{\kappa_4}{c^6} \sin 3\tilde{v}. \tag{13}$$

The coefficients κ_i are some PN accurate functions of E, h, and η. Since these expressions are lengthy and are only required temporarily, we do not list them in the paper.

Let us now turn our attention to the parametrization of the angular part, which is also required to obtain the final representation for the PN accurate Kepler equation. Using the 3PN accurate expression for $\dot{\phi}$ and the factorized 3PN accurate expression for $\dot{s}$, we compute $\frac{d\phi}{ds} = \dot{\phi}/\dot{s}$ and readily obtain the following relation between ϕ and s:

$$\phi - \phi_0 = \int_s^{s_-} \frac{B_0 + B_1\bar{s} + B_2\bar{s}^2 + B_3\bar{s}^3 + B_4\bar{s}^4 + B_5\bar{s}^5}{\sqrt{(s_- - \bar{s})(\bar{s} - s_+)}} d\bar{s}, \tag{14}$$

where the coefficients $B_i : i = 1$–5 are some PN functions of E, h, and η as presented in Appendix B. The next step involves the computation of the advance of periastron during the radial period P. This quantity, denoted by Φ, is obtained by evaluating the integral on the right-hand side of Eq. (14) between s_+ and s_-,

$$\Phi = 2 \int_{s_+}^{s_-} \frac{B_0 + B_1\bar{s} + B_2\bar{s}^2 + B_3\bar{s}^3 + B_4\bar{s}^4 + B_5\bar{s}^5}{\sqrt{(s_- - \bar{s})(\bar{s} - s_+)}} d\bar{s}. \tag{15}$$

To derive a temporary parametrization of the angular motion, using Eqs. (14) and (15), we first compute $\frac{2\pi}{\Phi} \times (\phi - \phi_0)$. Employing certain trigonometric relations, given in Appendix B, we arrive at a temporary 3PN accurate parametrization for the angular motion:

$$\frac{2\pi}{\Phi}(\phi - \phi_0) = \tilde{v} + \frac{\lambda_1}{c^2}\sin\tilde{v} + \frac{\lambda_2}{c^4}\sin 2\tilde{v} + \frac{\lambda_3}{c^4}\sin 3\tilde{v}$$
$$+ \frac{\lambda_4}{c^6}\sin 4\tilde{v} + \frac{\lambda_5}{c^6}\sin 5\tilde{v}, \tag{16}$$

where λ_i are some PN accurate functions, expressible in terms of E, h, and η.

To obtain the final parametrization for l and ϕ equations, we introduce $v = 2\arctan\{[(1 + e_\phi)/(1 - e_\phi)]^{1/2}\tan\frac{u}{2}\}$, where e_ϕ differs from e_r by yet to be determined PN corrections. It is straightforward to express $\tilde{v}$ in terms of v, some trigonometric functions of v, and the PN corrections relating e_ϕ and e_r, as shown schematically in Eq. (B2). We systematically introduce v in Eq. (16) such that the coefficient of $\sin v$ is zero up to the third PN order. This uniquely determines the PN corrections defining e_ϕ and consistently generalizes the imposed requirement that the expression for $\frac{2\pi}{\Phi}(\phi - \phi_0)$ at 1PN order has a Keplerian structure. However, we find that, at 2PN and 3PN orders, the v contribution will be supplemented by other trigonometric functions of v. With the introduction of v, the 3PN accurate expression for $\frac{2\pi}{\Phi}(\phi - \phi_0)$, given by Eq. (16), becomes

$$\frac{2\pi}{\Phi}(\phi - \phi_0) = v + \left(\frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6}\right)\sin 2v + \left(\frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6}\right)$$
$$\times \sin 3v + \frac{i_{6\phi}}{c^6}\sin 4v + \frac{h_{6\phi}}{c^6}\sin 5v, \tag{17}$$

where $v = 2\arctan\{[(1 + e_\phi)/(1 - e_\phi)]^{1/2}\tan\frac{u}{2}\}$. The explicit 3PN accurate expressions for Φ, e_ϕ, and the orbital functions will be displayed below.

Let us now go back to Eq. (13), giving the parametrization to l in terms of u, e_t, and $\tilde{v}(u)$. Using the 3PN accurate relation connecting $\tilde{v}$ and v, we rewrite that equation and obtain the following 3PN extension to the Kepler equation:

$$l \equiv n(t - t_0)$$
$$= u - e_t\sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)(v - u) + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right)$$
$$\times \sin v + \frac{i_{6t}}{c^6}\sin 2v + \frac{h_{6t}}{c^6}\sin 3v. \tag{18}$$

The 3PN accurate expressions for n, e_t, and the orbital functions appearing in the above Kepler equation will be displayed below.

Finally, we display, in its entirety, the third post-Newtonian accurate generalized quasi-Keplerian parametrization for a compact binary moving in an eccentric orbit in ADM-type coordinates:

$$r = a_r(1 - e_r\cos u), \tag{19a}$$

$$l \equiv n(t - t_0) = u - e_t\sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)(v - u) + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right)\sin v + \frac{i_{6t}}{c^6}\sin 2v + \frac{h_{6t}}{c^6}\sin 3v, \tag{19b}$$

$$\frac{2\pi}{\Phi}(\phi - \phi_0) = v + \left(\frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6}\right)\sin 2v + \left(\frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6}\right)\sin 3v + \frac{i_{6\phi}}{c^6}\sin 4v + \frac{h_{6\phi}}{c^6}\sin 5v, \tag{19c}$$

where $v = 2\arctan\{[(1 + e_\phi)/(1 - e_\phi)]^{1/2}\tan\frac{u}{2}\}$. The 3PN accurate expressions for the orbital elements a_r, e_r^2, n, e_t^2, Φ, and e_ϕ^2 and the post-Newtonian orbital functions g_{4t}, g_{6t}, f_{4t}, f_{6t}, i_{6t}, h_{6t}, $f_{4\phi}$, $f_{6\phi}$, $g_{4\phi}$, $g_{6\phi}$, $i_{6\phi}$, and $h_{6\phi}$, in terms of E, h, and η read

$$a_r = \frac{1}{(-2E)}\left\{1 + \frac{(-2E)}{4c^2}(-7+\eta) + \frac{(-2E)^2}{16c^4}\left[(1+10\eta+\eta^2) + \frac{1}{(-2Eh^2)}(-68+44\eta)\right]\right.$$

$$+ \frac{(-2E)^3}{192c^6}\left[3 - 9\eta - 6\eta^2 + 3\eta^3 + \frac{1}{(-2Eh^2)}[864 + (-3\pi^2 - 2212)\eta + 432\eta^2]\right.$$

$$\left.\left. + \frac{1}{(-2Eh^2)^2}[-6432 + (13488 - 240\pi^2)\eta - 768\eta^2]\right]\right\}, \tag{20a}$$

$$e_r^2 = 1 + 2Eh^2 + \frac{(-2E)}{4c^2}\{24 - 4\eta + 5(-3+\eta)(-2Eh^2)\} + \frac{(-2E)^2}{8c^4}\left\{52 + 2\eta + 2\eta^2 - (80 - 55\eta + 4\eta^2)(-2Eh^2)\right.$$

$$\left. - \frac{8}{(-2Eh^2)}(-17+11\eta)\right\} + \frac{(-2E)^3}{192c^6}\left\{-768 - 6\eta\pi^2 - 344\eta - 216\eta^2 + 3(-2Eh^2)(-1488 + 1556\eta - 319\eta^2\right.$$

$$\left. + 4\eta^3) - \frac{4}{(-2Eh^2)}(588 - 8212\eta + 177\eta\pi^2 + 480\eta^2) + \frac{192}{(-2Eh^2)^2}(134 - 281\eta + 5\eta\pi^2 + 16\eta^2)\right\}, \tag{20b}$$

$$n = (-2E)^{3/2}\left\{1 + \frac{(-2E)}{8c^2}(-15+\eta) + \frac{(-2E)^2}{128c^4}\left[555 + 30\eta + 11\eta^2 + \frac{192}{\sqrt{(-2Eh^2)}}(-5+2\eta)\right]\right.$$

$$+ \frac{(-2E)^3}{3072c^6}\left[-29\,385 - 4995\eta - 315\eta^2 + 135\eta^3 - \frac{16}{(-2Eh^2)^{3/2}}(10\,080 + 123\eta\pi^2 - 13\,952\eta + 1440\eta^2)\right.$$

$$\left.\left. + \frac{5760}{\sqrt{(-2Eh^2)}}(17 - 9\eta + 2\eta^2)\right]\right\}, \tag{20c}$$

$$e_t^2 = 1 + 2Eh^2 + \frac{(-2E)}{4c^2}\{-8 + 8\eta - (-17+7\eta)(-2Eh^2)\} + \frac{(-2E)^2}{8c^4}\left\{8 + 4\eta + 20\eta^2 - (-2Eh^2)(112 - 47\eta + 16\eta^2)\right.$$

$$\left. - 24\sqrt{(-2Eh^2)}(-5+2\eta) + \frac{4}{(-2Eh^2)}(17 - 11\eta) - \frac{24}{\sqrt{(-2Eh^2)}}(5-2\eta)\right\} + \frac{(-2E)^3}{192c^6}\left\{24(-2+5\eta)\right.$$

$$\times(-23 + 10\eta + 4\eta^2) - 15(-528 + 200\eta - 77\eta^2 + 24\eta^3)(-2Eh^2) - 72(265 - 193\eta + 46\eta^2)\sqrt{(-2Eh^2)}$$

$$- \frac{2}{(-2Eh^2)}(6732 + 117\eta\pi^2 - 12\,508\eta + 2004\eta^2) + \frac{2}{\sqrt{(-2Eh^2)}}(16\,380 - 19\,964\eta + 123\eta\pi^2 + 3240\eta^2)$$

$$\left. - \frac{2}{(-2Eh^2)^{3/2}}(10\,080 + 123\eta\pi^2 - 13\,952\eta + 1440\eta^2) + \frac{96}{(-2Eh^2)^2}(134 - 281\eta + 5\eta\pi^2 + 16\eta^2)\right\}, \tag{20d}$$

$$g_{4t} = \frac{3(-2E)^2}{2}\left\{\frac{5-2\eta}{\sqrt{(-2Eh^2)}}\right\}, \tag{20e}$$

$$g_{6t} = \frac{(-2E)^3}{192}\left\{\frac{1}{(-2Eh^2)^{3/2}}(10\,080 + 123\eta\pi^2 - 13\,952\eta + 1440\eta^2) + \frac{1}{\sqrt{(-2Eh^2)}}(-3420 + 1980\eta - 648\eta^2)\right\}, \tag{20f}$$

$$f_{4t} = -\frac{1}{8}\frac{(-2E)^2}{\sqrt{(-2Eh^2)}}\{(4+\eta)\eta\sqrt{(1+2Eh^2)}\}, \tag{20g}$$

$$f_{6t} = \frac{(-2E)^3}{192}\left\{\frac{1}{(-2Eh^2)^{3/2}}\frac{1}{\sqrt{1+2Eh^2}}(1728 - 4148\eta + 3\eta\pi^2 + 600\eta^2 + 33\eta^3) + 3\frac{\sqrt{(-2Eh^2)}}{\sqrt{(1+2Eh^2)}}\right.$$

$$\left. \times\eta(-64 - 4\eta + 23\eta^2) + \frac{1}{\sqrt{(-2Eh^2)(1+2Eh^2)}}(-1728 + 4232\eta - 3\eta\pi^2 - 627\eta^2 - 105\eta^3)\right\}, \tag{20h}$$

$$i_{6t} = \frac{(-2E)^3}{32}\eta\left\{\frac{(1+2Eh^2)}{(-2Eh^2)^{3/2}}(23 + 12\eta + 6\eta^2)\right\}, \tag{20i}$$

$$h_{6t} = \frac{13(-2E)^3}{192}\eta^3\left(\frac{1+2Eh^2}{-2Eh^2}\right)^{3/2},\tag{20j}$$

$$\Phi = 2\pi\left\{1+\frac{3}{c^2h^2}+\frac{(-2E)^2}{4c^4}\left[\frac{3}{(-2Eh^2)}(-5+2\eta)+\frac{15}{(-2Eh^2)^2}(7-2\eta)\right]+\frac{(-2E)^3}{128c^6}\left[\frac{24}{(-2Eh^2)}(5-5\eta+4\eta^2)\right.\right.$$
$$\left.\left.-\frac{1}{(-2Eh^2)^2}(10\,080-13\,952\eta+123\eta\pi^2+1440\eta^2)+\frac{5}{(-2Eh^2)^3}(7392-8000\eta+123\eta\pi^2+336\eta^2)\right]\right\},\tag{20k}$$

$$f_{4\phi}=\frac{(-2E)^2}{8}\frac{(1+2Eh^2)}{(-2Eh^2)^2}\eta(1-3\eta),\tag{20l}$$

$$f_{6\phi}=\frac{(-2E)^3}{256}\left\{\frac{4\eta}{(-2Eh^2)}(-11-40\eta+24\eta^2)+\frac{1}{(-2Eh^2)^2}(-256+1192\eta-49\eta\pi^2+336\eta^2-80\eta^3)\right.$$
$$\left.+\frac{1}{(-2Eh^2)^3}(256+49\eta\pi^2-1076\eta-384\eta^2-40\eta^3)\right\},\tag{20m}$$

$$g_{4\phi}=-\frac{3(-2E)^2}{32}\frac{\eta^2}{(-2Eh^2)^2}(1+2Eh^2)^{3/2},\tag{20n}$$

$$g_{6\phi}=\frac{(-2E)^3}{768}\sqrt{(1+2Eh^2)}\left\{-\frac{3}{(-2Eh^2)}\eta^2(9-26\eta)-\frac{1}{(-2Eh^2)^2}\eta(220+3\pi^2+312\eta+150\eta^2)\right.$$
$$\left.+\frac{1}{(-2Eh^2)^3}\eta(220+3\pi^2+96\eta+45\eta^2)\right\},\tag{20o}$$

$$i_{6\phi}=\frac{(-2E)^3}{128}\frac{(1+2Eh^2)^2}{(-2Eh^2)^3}\eta(5+28\eta+10\eta^2),\tag{20p}$$

$$h_{6\phi}=\frac{5(-2E)^3}{256}\frac{\eta^3}{(-2Eh^2)^3}(1+2Eh^2)^{5/2},\tag{20q}$$

$$e_\phi^2=1+2Eh^2+\frac{(-2E)}{4c^2}\{24+(-15+\eta)(-2Eh^2)\}+\frac{(-2E)^2}{16c^4}\left\{-32+176\eta+18\eta^2-(-2Eh^2)(160-30\eta+3\eta^2)\right.$$
$$\left.+\frac{1}{(-2Eh^2)}(408-232\eta-15\eta^2)\right\}+\frac{(-2E)^3}{384c^6}\left\{-16\,032+2764\eta+3\eta\pi^2+4536\eta^2+234\eta^3\right.$$
$$-36(248-80\eta+13\eta^2+\eta^3)(-2Eh^2)-\frac{6}{(-2Eh^2)}(2456-26\,860\eta+581\eta\pi^2+2689\eta^2+10\eta^3)$$
$$\left.+\frac{3}{(-2Eh^2)^2}(27\,776-65\,436\eta+1325\eta\pi^2+3440\eta^2-70\eta^3)\right\}.\tag{20r}$$

The three eccentricities e_r, e_t and e_ϕ, which differ from each other at PN orders, are related by

$$e_t=e_r\left\{1+\frac{(-2E)}{2c^2}(-8+3\eta)+\frac{(-2E)^2}{8c^4}\frac{1}{(-2Eh^2)}[-34+22\eta+(-60+24\eta)\sqrt{(-2Eh^2)}+(72-33\eta+12\eta^2)\right.$$
$$\times(-2Eh^2)]+\frac{(-2E)^3}{192c^6}\frac{1}{(-2Eh^2)^2}[-6432+13\,488\eta-240\eta\pi^2-768\eta^2+(-10\,080+13\,952\eta-123\eta\pi^2-1440\eta^2)$$
$$\times\sqrt{(-2Eh^2)}+(2700-4420\eta-3\eta\pi^2+1092\eta^2)(-2Eh^2)+(9180-6444\eta+1512\eta^2)(-2Eh^2)^{3/2}$$
$$\left.+(-3840+1284\eta-672\eta^2+240\eta^3)(-2Eh^2)^2]\right\},\tag{21a}$$

$$e_\phi=e_r\left\{1+\frac{(-2E)}{2c^2}\eta+\frac{(-2E)^2}{32c^4}\frac{1}{(-2Eh^2)}[136-56\eta-15\eta^2+\eta(20+11\eta)(-2Eh^2)]+\frac{(-2E)^3}{768c^6}\frac{1}{(-2Eh^2)^2}\right.$$
$$\times[31\,872-88\,404\eta+2055\eta\pi^2+4176\eta^2-210\eta^3+(2256+10\,228\eta-15\eta\pi^2-2406\eta^2-450\eta^3)$$
$$\left.(-2Eh^2)+6\eta(136+34\eta+31\eta^2)(-2Eh^2)^2]\right\}.\tag{21b}$$

These relations allow one to choose a specific eccentricity, while describing a PN accurate noncircular orbit.

It is highly desirable that such a detailed parametrization and lengthy expressions for the PN accurate orbital elements and functions should be subjected to possible consistency checks. We have devised and performed the following consistency check for our computation. Note that the expressions for a_r and e_r^2 were obtained from the PN accurate roots s_- and s_+, representing the turning points of the radial motion. This prompted us to express the expressions for $\dot{r}^2$ and $\dot{\phi}^2$, derived using the Hamiltonian equations of motion and given by Eqs. (8a) and (8b), in terms of E, h, η, and $[1 - e_r \cos(u)]$. We then compared the above expression for $\dot{r}^2$ with the one derived using the parametrization, namely $\dot{r}^2 = (\frac{dr}{du}\frac{du}{dt})^2$. This expression for $\dot{r}^2$, after some lengthy algebra, is found to be in total agreement with $\dot{r}^2$, computed using Hamiltonian equations of motion to the third post-Newtonian order. We note that the above computation fully checked both the structure and parameters of both the radial and temporal part of the generalized quasi-Keplerian representation. We performed a similar check on the angular part by noting that $\dot{\phi}^2 = (\frac{d\phi}{dv}\frac{dv}{du}\frac{du}{dt})^2$. The expressions for $\dot{\phi}^2$, computed using the above relation and via the Hamiltonian equations of motion, were also found to be in total agreement to 3PN order. These two computations provided us with powerful checks on our 3PN accurate generalized quasi-Keplerian parametrization.

Finally, we observe that, to the 2PN order, our results are in agreement with results available in [14,15]. The above parametrization is also consistent with results given in [16], modulo typographical errors and omissions. In the next section, we obtain a similar parametrization in harmonic gauge.

IV. THE 3PN ACCURATE GENERALIZED QUASI-KEPLERIAN PARAMETRIZATION IN HARMONIC COORDINATES

Incidentally, it was in harmonic coordinates that the quasi-Keplerian parametrization for compact binaries of arbitrary mass ratio moving in eccentric orbits was first realized [11]. The above computation employed 1PN accurate expressions for the conserved orbital energy and angular momentum in terms of the dynamical variables of the binary to derive 1PN accurate expressions, in harmonic coordinates, for $\dot{r}^2$ and $\dot{\phi}^2$. However, at 2PN order, the parametrization was achieved only in ADM gauge [14,15], though the description of the binary dynamics, in these two gauges, begins to differ at 2PN order [20].

In this section, we will compute 3PN accurate generalized quasi-Keplerian parametrization, in har-

monic coordinates, for compact binaries in eccentric orbits. To begin our computation, we will require 3PN accurate expressions for the conserved orbital energy and angular momentum in harmonic coordinates and in the center-of-mass frame. These quantities, written in terms of the dynamical variables of the binary, are required to derive, by iteration, 3PN accurate expressions for $\dot{r}^2$ and $\dot{\phi}^2$ in terms of E, h, and r. The above-mentioned 3PN accurate conserved orbital energy and angular momentum, in harmonic coordinates, are available in [25]. (For a comprehensive review of the post-Newtonian computations, especially in harmonic coordinates, see [26].) However, we will not be able to employ directly these conserved quantities in our computation for $\dot{r}^2$ and $\dot{\phi}^2$ for the following two reasons. First, at 3PN order, these expressions contain logarithmic terms involving the radial separation r. These logarithmic terms will not allow us to obtain 3PN accurate expressions for $\dot{r}^2$ and $\dot{\phi}^2$ as polynomials in $s = 1/r$, preventing the determination of 3PN accurate roots of $\dot{r}^2$ and, hence, the parametrization in harmonic coordinates. However, in the near zone of a gravitating system, the harmonic gauge conditions do not fix the coordinate system uniquely, allowing various harmonic coordinates to be employed at 3PN order [27–30]. Further, it is possible to remove these logarithmic terms using a 3PN accurate coordinate transformation that still respects the harmonic gauge condition [28]. The problematic logarithmic terms appearing in the 3PN accurate expressions for the orbital energy and angular momentum, as available in [25], were removed using the coordinate transformation given in [28]. Second, the expression for 3PN accurate orbital energy appearing in [25] involves an undetermined parameter, the so-called regularization ambiguity, which was recently fixed by different techniques [23,30,31]. These additional steps allowed the compilation of the following 3PN accurate expressions for the orbital energy and angular momentum, in harmonic gauge and in the center-of-mass frame, reduced by μ as

$$E = E_0 + \frac{1}{c^2}E_1 + \frac{1}{c^4}E_2 + \frac{1}{c^6}E_3, \tag{22a}$$

$$J = |\mathbf{R} \times \mathbf{V}|\left\{J_0 + \frac{1}{c^2}J_1 + \frac{1}{c^4}J_2 + \frac{1}{c^6}J_3\right\}, \tag{22b}$$

where $\mathbf{R}$ and $\mathbf{V}$ are the relative separation and velocity vectors. The various contributions to E and J are given by

$$E_0 = \frac{1}{2}v^2 - \frac{1}{r}, \tag{23a}$$

$$E_1 = \frac{3}{8}(1 - 3\eta)v^4 + \frac{1}{2}(3 + \eta)\frac{v^2}{r} + \frac{1}{2}\eta\frac{\dot{r}^2}{r} + \frac{1}{2}\left(\frac{1}{r}\right)^2, \tag{23b}$$

$$E_2 = \frac{5}{16}(1 - 7\eta + 13\eta^2)v^6 + \frac{1}{8}(21 - 23\eta - 27\eta^2)\frac{v^4}{r} + \frac{1}{4}\eta(1 - 15\eta)\frac{v^2\dot{r}^2}{r} - \frac{3}{8}\eta(1 - 3\eta)\frac{\dot{r}^4}{r}$$
$$+ \frac{1}{8}(14 - 55\eta + 4\eta^2)\left(\frac{v}{r}\right)^2 + \frac{1}{8}(4 + 69\eta + 12\eta^2)\left(\frac{\dot{r}}{r}\right)^2 - \frac{1}{4}(2 + 15\eta)\left(\frac{1}{r}\right)^3, \tag{23c}$$

$$E_3 = \frac{1}{128}(35 - 413\eta + 1666\eta^2 - 2261\eta^3)v^8 + \frac{1}{16}(55 - 215\eta + 116\eta^2 + 325\eta^3)\frac{v^6}{r} + \frac{1}{16}\eta(5 - 25\eta + 25\eta^2)$$
$$\times \frac{\dot{r}^6}{r} - \frac{1}{16}\eta(21 + 75\eta - 375\eta^2)\frac{v^4\dot{r}^2}{r} - \frac{1}{16}\eta(9 - 84\eta + 165\eta^2)\frac{v^2\dot{r}^4}{r} + \frac{1}{16}(135 - 194\eta + 406\eta^2 - 108\eta^3)$$
$$\times \left(\frac{v^2}{r}\right)^2 + \frac{1}{16}(12 + 248\eta - 815\eta^2 - 324\eta^3)\left(\frac{v\dot{r}}{r}\right)^2 - \frac{1}{48}\eta(731 - 492\eta - 288\eta^2)\left(\frac{\dot{r}^2}{r}\right)^2$$
$$+ \frac{1}{2240}[2800 - (53\,976 - 1435\pi^2)\eta - 11\,760\eta^2 + 1120\eta^3]\frac{v^2}{r^3} + \frac{1}{2240}[3360 + (18\,568 - 4305\pi^2)$$
$$\times \eta + 28\,560\eta^2 + 7840\eta^3]\frac{\dot{r}^2}{r^3} + \frac{1}{840}(315 + 18\,469\eta)\left(\frac{1}{r}\right)^4, \tag{23d}$$

$$J_0 = 1, \tag{23e}$$

$$J_1 = \frac{1}{2}(1 - 3\eta)v^2 + (3 + \eta)\frac{1}{r}, \tag{23f}$$

$$J_2 = \frac{3}{8}(1 - 7\eta + 13\eta^2)v^4 + \frac{1}{2}(7 - 10\eta - 9\eta^2)\frac{v^2}{r} - \frac{1}{2}\eta(2 + 5\eta)\frac{\dot{r}^2}{r} + \frac{1}{4}(14 - 41\eta + 4\eta^2)\left(\frac{1}{r}\right)^2, \tag{23g}$$

$$J_3 = \frac{1}{16}(5 - 59\eta + 238\eta^2 - 323\eta^3)v^6 + \frac{1}{8}(33 - 142\eta + 106\eta^2 + 195\eta^3)\frac{v^4}{r} - \frac{1}{4}\eta(12 - 7\eta - 75\eta^2)\frac{v^2\dot{r}^2}{r}$$
$$+ \frac{3}{8}\eta(2 - 2\eta - 11\eta^2)\frac{\dot{r}^4}{r} + \frac{1}{12}(135 - 322\eta + 315\eta^2 - 108\eta^3)\left(\frac{v}{r}\right)^2 + \frac{1}{24}(12 - 287\eta - 951\eta^2 - 324\eta^3)$$
$$\times \left(\frac{\dot{r}}{r}\right)^2 + \left[\frac{5}{2} - \frac{1}{1120}(20\,796 - 1435\pi^2)\eta - 7\eta^2 + \eta^3\right]\left(\frac{1}{r}\right)^3. \tag{23h}$$

In the above expressions, $r = |\mathbf{r}|$ with $\mathbf{r} = \mathbf{R}/(GM)$, $v^2 = \mathbf{v} \cdot \mathbf{v}$, and $\dot{r} = \frac{\mathbf{v} \cdot \mathbf{r}}{r}$, where $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ and t is the coordinate time scaled by GM. The above expressions also match those presented in [8].

Since the orbital motion is restricted to a plane, we express the components of $\mathbf{r}$ in polar coordinates as $\mathbf{r} = r(\cos\phi, \sin\phi)$ and obtain $\mathbf{v} = (\dot{r}\cos\phi - r\dot{\phi}\sin\phi, \dot{r}\sin\phi + r\dot{\phi}\cos\phi)$, allowing us to write $v^2 = \dot{r}^2 + r^2\dot{\phi}^2$. Using these inputs and 3PN accurate expressions for E and J, given by Eqs. (22) and (23), we obtain, after lengthy iterations, 3PN accurate expressions for $\dot{r}^2$ and $\dot{\phi}$, in terms of E, $h = \frac{J}{GM}$, η, and r. This leads to 3PN accurate expressions for $\dot{r}^2$ and $(d\phi/ds)$, in harmonic coordinates, displayed in Appendix A. We note that the expressions for $\dot{r}^2$ and $\dot{\phi}$ differ from similar ones derived for ADM-type gauge at 2PN and 3PN orders. However, note that in both gauges the ex-

pressions for $\dot{r}^2$ and $\dot{\phi}$ are polynomials of degree seven. It is interesting to note that in the previous section the orbital dynamics was fully determined with the help of the Hamiltonian. However, here we employed neither the Hamiltonian nor the Lagrangian to determine the orbital dynamics. Though the energy is numerically equal to the Hamiltonian, we required both the energy and angular momentum as functions of particle positions and velocities to determine the binary dynamics. The analogy to energy as thermodynamical potential versus caloric and thermal equations of state is quite appropriate [32].

Following exactly the same procedure detailed in the previous section, we derive the 3PN accurate orbital parametrization in harmonic gauge. The third post-Newtonian accurate generalized quasi-Keplerian parametrization, in harmonic coordinates, for compact binaries moving in eccentric orbits is given by

$$r = a_r(1 - e_r \cos u), \tag{24a}$$

$$l \equiv n(t - t_0) = u - e_t \sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)(v - u) + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right)\sin v + \frac{i_{6t}}{c^6}\sin 2v + \frac{h_{6t}}{c^6}\sin 3v, \tag{24b}$$

$$\frac{2\pi}{\Phi}(\phi - \phi_0) = v + \left(\frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6}\right)\sin 2v + \left(\frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6}\right)\sin 3v + \frac{i_{6\phi}}{c^6}\sin 4v + \frac{h_{6\phi}}{c^6}\sin 5v, \tag{24c}$$

where $v = 2 \arctan\{[(1 + e_\phi)/(1 - e_\phi)]^{1/2} \tan\frac{u}{2}\}$. The explicit 3PN accurate expressions for the orbital elements and functions of the generalized quasi-Keplerian parametrization, in harmonic coordinates, read

$$a_r = \frac{1}{(-2E)}\left\{1 + \frac{(-2E)}{4c^2}(-7 + \eta) + \frac{(-2E)^2}{16c^4}\left[1 + \eta^2 + \frac{16}{(-2Eh^2)}(-4 + 7\eta)\right]\right.$$
$$+ \frac{(-2E)^3}{6720c^6}\left[105 - 105\eta + 105\eta^3 + \frac{1}{(-2Eh^2)}(26\,880 + 4305\pi^2\eta - 215\,408\eta + 47\,040\eta^2)\right.$$
$$\left.\left. - \frac{4}{(-2Eh^2)^2}(53\,760 - 176\,024\eta + 4305\pi^2\eta + 15\,120\eta^2)\right]\right\}, \tag{25a}$$

$$e_r^2 = 1 + 2Eh^2 + \frac{(-2E)}{4c^2}\{24 - 4\eta + 5(-3 + \eta)(-2Eh^2)\} + \frac{(-2E)^2}{8c^4}\left\{60 + 148\eta + 2\eta^2 - (-2Eh^2)(80 - 45\eta + 4\eta^2)\right.$$
$$\left. + \frac{32}{(-2Eh^2)}(4 - 7\eta)\right\} + \frac{(-2E)^3}{6720c^6}\left\{-3360 + 181\,264\eta + 8610\pi^2\eta - 67\,200\eta^2 + 105(-2Eh^2)\right.$$
$$\times(-1488 + 1120\eta - 195\eta^2 + 4\eta^3) - \frac{80}{(-2Eh^2)}(1008 - 21\,130\eta + 861\pi^2\eta + 2268\eta^2)$$
$$\left. + \frac{16}{(-2Eh^2)^2}(53\,760 - 176\,024\eta + 4305\pi^2\eta + 15\,120\eta^2)\right\}, \tag{25b}$$

$$n = (-2E)^{3/2}\left\{1 + \frac{(-2E)}{8c^2}(-15 + \eta) + \frac{(-2E)^2}{128c^4}\left[555 + 30\eta + 11\eta^2 + \frac{192}{\sqrt{(-2Eh^2)}}(-5 + 2\eta)\right]\right.$$
$$+ \frac{(-2E)^3}{3072c^6}\left[-29\,385 - 4995\eta - 315\eta^2 + 135\eta^3 + \frac{5760}{\sqrt{(-2Eh^2)}}(17 - 9\eta + 2\eta^2) - \frac{16}{(-2Eh^2)^{3/2}}\right.$$
$$\left.\left.\times(10\,080 - 13\,952\eta + 123\pi^2\eta + 1440\eta^2)\right]\right\}, \tag{25c}$$

$$e_t^2 = 1 + 2Eh^2 + \frac{(-2E)}{4c^2}\{-8 + 8\eta - (-2Eh^2)(-17 + 7\eta)\} + \frac{(-2E)^2}{8c^4}\left\{12 + 72\eta + 20\eta^2 - 24\sqrt{(-2Eh^2)}(-5 + 2\eta)\right.$$
$$\left. - (-2Eh^2)(112 - 47\eta + 16\eta^2) - \frac{16}{(-2Eh^2)}(-4 + 7\eta) + \frac{24}{\sqrt{(-2Eh^2)}}(-5 + 2\eta)\right\}$$
$$+ \frac{(-2E)^3}{6720c^6}\left\{23\,520 - 464\,800\eta + 179\,760\eta^2 + 16\,800\eta^3 - 2520\sqrt{(-2Eh^2)}(265 - 193\eta + 46\eta^2)\right.$$
$$- 525(-2Eh^2)(-528 + 200\eta - 77\eta^2 + 24\eta^3) - \frac{6}{(-2Eh^2)}(73\,920 - 26\,0272\eta + 4305\pi^2\eta + 61\,040\eta^2)$$
$$+ \frac{70}{\sqrt{(-2Eh^2)}}(16\,380 - 19\,964\eta + 123\pi^2\eta + 3240\eta^2) + \frac{8}{(-2Eh^2)^2}(53\,760 - 176\,024\eta + 4305\pi^2\eta$$
$$\left. + 15\,120\eta^2) - \frac{70}{(-2Eh^2)^{3/2}}(10\,080 - 13\,952\eta + 123\pi^2\eta + 1440\eta^2)\right\}, \tag{25d}$$

$$g_{4t} = -\frac{3(-2E)^2}{2}\left\{\frac{1}{\sqrt{(-2Eh^2)}}(-5 + 2\eta)\right\}, \tag{25e}$$

$$g_{6t} = \frac{(-2E)^3}{192} \left\{ \frac{1}{(-2Eh^2)^{3/2}} (10\,080 - 13\,952\eta + 123\pi^2\eta + 1440\eta^2) + \frac{1}{\sqrt{(-2Eh^2)}} (-3420 + 1980\eta - 648\eta^2) \right\},$$

(25f)

$$f_{4t} = -\frac{(-2E)^2}{8} \left\{ \frac{\sqrt{1+2Eh^2}}{\sqrt{(-2Eh^2)}} \eta(-15 + \eta) \right\},$$

(25g)

$$f_{6t} = \frac{(-2E)^3}{2240} \left\{ \frac{1}{(-2Eh^2)^{3/2}\sqrt{1+2Eh^2}} (22\,400 + 43\,651\eta - 1435\pi^2\eta - 20\,965\eta^2 + 385\eta^3) + \frac{1}{\sqrt{(-2Eh^2)}\sqrt{1+2Eh^2}} \right.$$
$$\left. \times(-22\,400 - 49\,321\eta + 27\,300\eta^2 + 1435\pi^2\eta - 1225\eta^3) + \frac{35\sqrt{(-2Eh^2)}}{\sqrt{1+2Eh^2}} \eta(297 - 175\eta + 23\eta^2) \right\},$$

(25h)

$$i_{6t} = \frac{(-2E)^3}{16} \left\{ \frac{1+2Eh^2}{(-2Eh^2)^{3/2}} \eta(116 - 49\eta + 3\eta^2) \right\},$$

(25i)

$$h_{6t} = \frac{(-2E)^3}{192} \left\{ \left(\frac{1+2Eh^2}{(-2Eh^2)} \right)^{3/2} \eta(23 - 73\eta + 13\eta^2) \right\},$$

(25j)

$$\Phi = 2\pi \left\{ 1 + \frac{3}{h^2 c^2} + \frac{(-2E)^2}{4c^4} \left[\frac{3}{(-2Eh^2)} (-5 + 2\eta) - \frac{15}{(-2Eh^2)^2} (-7 + 2\eta) \right] \right.$$
$$+ \frac{(-2E)^3}{128c^6} \left[\frac{5}{(-2Eh^2)^3} (7392 - 8000\eta + 336\eta^2 + 123\pi^2\eta) + \frac{24}{(-2Eh^2)} (5 - 5\eta + 4\eta^2) \right.$$
$$\left.\left. - \frac{1}{(-2Eh^2)^2} (10\,080 - 13\,952\eta + 123\pi^2\eta + 1440\eta^2) \right] \right\},$$

(25k)

$$f_{4\phi} = \frac{(-2E)^2}{8} \left\{ \frac{1+2Eh^2}{(-2Eh^2)^2} (1 + 19\eta - 3\eta^2) \right\},$$

(25l)

$$f_{6\phi} = \frac{(-2E)^3}{26880} \left\{ \frac{1}{(-2Eh^2)^3} (67\,200 + 994\,704\eta - 30\,135\pi^2\eta - 335\,160\eta^2 - 4200\eta^3) + \frac{1}{(-2Eh^2)^2} \right.$$
$$\times(-60\,480 - 991\,904\eta + 30\,135\pi^2\eta + 428\,400\eta^2 - 8400\eta^3) + \frac{1}{(-2Eh^2)}$$
$$\left. \times(840 + 141\,680\eta - 99\,960\eta^2 + 10\,080\eta^3) \right\},$$

(25m)

$$g_{4\phi} = -\frac{(-2E)^2}{32} \left\{ \frac{(1+2Eh^2)^{3/2}}{(-2Eh^2)^2} \eta(-1 + 3\eta) \right\},$$

(25n)

$$g_{6\phi} = \frac{(-2E)^3}{8960} \frac{1}{\sqrt{1+2Eh^2}} \left\{ -35\eta(14 - 49\eta + 26\eta^2) - \frac{1}{(-2Eh^2)} \eta(-36\,196 + 1435\pi^2 + 29\,225\eta - 2660\eta^2) \right.$$
$$+ \frac{1}{(-2Eh^2)^2} \eta(-71\,867 + 2870\pi^2 + 56\,035\eta - 2275\eta^2) - \frac{1}{(-2Eh^2)^3} \eta(-36\,161 + 1435\pi^2$$
$$\left. + 28\,525\eta - 525\eta^2) \right\},$$

(25o)

$$i_{6\phi} = \frac{(-2E)^3}{192} \left\{ \frac{(1+2Eh^2)^2}{(-2Eh^2)^3} \eta(82 - 57\eta + 15\eta^2) \right\},$$

(25p)

$$h_{6\phi} = \frac{(-2E)^3}{256} \left\{ \frac{(1+2Eh^2)^{5/2}}{(-2Eh^2)^3} \eta(1 - 5\eta + 5\eta^2) \right\},$$

(25q)

$$e_\phi^2 = 1 + 2Eh^2 + \frac{(-2E)}{4c^2}\{24 + (-2Eh^2)(-15 + \eta)\} + \frac{(-2E)^2}{16c^4}\Big\{-40 + 34\eta + 18\eta^2 - (-2Eh^2)(160 - 31\eta + 3\eta^2)$$

$$-\frac{1}{(-2Eh^2)}(-416 + 91\eta + 15\eta^2)\Big\} + \frac{(-2E)^3}{13440c^6}\Big\{-584\,640 - 17\,482\eta - 4305\pi^2\eta - 7350\eta^2 + 8190\eta^3$$

$$-420(-2Eh^2)(744 - 248\eta + 31\eta^2 + 3\eta^3) - \frac{14}{(-2Eh^2)}(36\,960 - 341\,012\eta + 4305\pi^2\eta - 225\eta^2 + 150\eta^3)$$

$$-\frac{1}{(-2Eh^2)^2}(-2\,956\,800 + 5\,627\,206\eta - 81\,795\pi^2\eta + 14\,490\eta^2 + 7350\eta^3)\Big\}. \tag{25r}$$

In harmonic coordinates also, there are PN accurate relations connecting the three eccentricities e_r, e_t, and e_ϕ. These relations read

$$e_t = e_r\Big\{1 + \frac{(-2E)}{2c^2}(3\eta - 8) + \frac{(-2E)^2}{4c^4}\frac{1}{(-2Eh^2)}[-16 + 28\eta + (-30 + 12\eta)\sqrt{(-2Eh^2)} + (36 - 19\eta + 6\eta^2)$$

$$\times(-2Eh^2)] + \frac{(-2E)^3}{6720c^6}\frac{1}{h^4(-2E)^2}[-215\,040 + 704\,096\eta - 17\,220\pi^2\eta - 60\,480\eta^2 + 35(-10\,080 + 13\,952\eta$$

$$-123\pi^2\eta - 1440\eta^2)\sqrt{(-2Eh^2)} + (87\,360 - 354\,848\eta + 4305\pi^2\eta + 105\,840\eta^2)(-2Eh^2) + (-134\,400 + 54\,600\eta$$

$$-28\,560\eta^2 + 8400\eta^3)(-2Eh^2)^2 + (321\,300 - 225\,540\eta + 52\,920\eta^2)(-2Eh^2)^{3/2}]\Big\}, \tag{26a}$$

$$e_\phi = e_r\Big\{1 + \frac{(-2E)}{2c^2}\eta + \frac{(-2E)^2}{32c^4}\frac{1}{(-2Eh^2)}[160 + 357\eta - 15\eta^2 + (-\eta + 11\eta^2)(-2Eh^2)] + \frac{(-2E)^3}{8960c^6}\frac{1}{(-2Eh^2)^2}$$

$$\times[412\,160 + 1854\eta - 18\,655\pi^2\eta - 166\,110\eta^2 - 2450\eta^3 + (24\,640 - 182\,730\eta + 7175\pi^2\eta$$

$$+156\,520\eta^2 - 5250\eta^3)(-2Eh^2) + 70\eta(-1 - \eta + 31\eta^2)(-2Eh^2)^2]\Big\}. \tag{26b}$$

We observe that the structure of the 3PN accurate parametrization is identical in both ADM-type and harmonic coordinates. This is not surprising as the expressions for $\dot{r}^2$ and $\dot{\phi}$ are polynomials of the same degree in $s = \frac{1}{r}$ in these two gauges. We also observe that since the equations of motion, and hence the expressions for $\dot{r}^2$ and $\dot{\phi}$, are the same in ADM and harmonic gauges to 1PN order, the orbital elements and the relations between the

eccentricities to 1PN order are also identical in these two gauges. It is also possible to connect the radial eccentricities, and hence other eccentricities, associated with parametrizations in ADM-like and harmonic coordinates. This indicates that a circular orbit in ADM-type gauge also implies a circular orbit in harmonic gauge. The relation connecting e_r^2 in ADM-like coordinates to that in harmonic coordinates reads

$$e_r^2|_{\rm H} = e_r^2|_{\rm A}\Big\{1 + \frac{(-2E)^2}{4c^4}\Big[5\eta + \frac{2}{Eh^2}(17\eta + 1)\Big] + \frac{(-2E)^3}{1680c^6}\Big[3570\eta - 630\eta^2 + \frac{1}{(-2Eh^2)}[420 - 23\,520\eta^2$$

$$+(72\,844 - 2205\pi^2)\eta] + \frac{4}{(-2Eh^2)^2}[-2520 + 8400\eta^2 - (58\,004 - 2205\pi^2)\eta]\Big]\Big\}, \tag{27}$$

where $e_r^2|_{\rm H}$ and $e_r^2|_{\rm A}$ are the expressions e_r^2 in harmonic and ADM-type coordinates.

The most striking result is that the expressions for n and Φ in terms of E and h are identical to 3PN order in both ADM-type and harmonic coordinates. Indeed it was shown, using the Hamilton-Jacobi approach to describe the relative trajectory of the binary, that the functional form of n and Φ should be independent of the coordinate system used, if expressed in terms of gauge (coordinate) invariant quantities such as E and h [14]. This should be contrasted with the functional forms for other orbital

elements such as a_r and e_r, in terms of E and h, which depend on the coordinate system used. The explicit verification of the above prediction constitutes a powerful check on the algebra involved in the derivation of 3PN accurate orbital representation in these two gauges.

The gauge invariance of n and Φ allows us to obtain the following gauge invariant expressions for the third post-Newtonian accurate adimensional orbital energy and angular momentum for compact binaries of arbitrary mass ratio, moving in eccentric orbits:

$$\frac{E}{c^2} = -\frac{x}{2}\left\{1 + \frac{1}{12}[15 - \eta]x + \frac{1}{24}\left[(120 - 48\eta)\left(\frac{k'}{x}\right)^{1/2} + 15 - 15\eta - \eta^2\right]x^2 + \frac{1}{5184}\left[[68\,040 + (-173\,376 + 2214\pi^2)\right.\right.$$

$$\left.\left. \times \eta + 12\,960\eta^2]\left(\frac{k'}{x}\right)^{3/2} + (68\,040 - 30\,240\eta - 6048\eta^2)\left(\frac{k'}{x}\right)^{1/2} - 4995 - 6075\eta - 450\eta^2 - 35\eta^3\right]x^3\right\}, \quad (28a)$$

$$(hc)^2 = \frac{1}{k'}\left\{1 + \frac{1}{4}\left[(35 - 10\eta)\left(\frac{k'}{x}\right) - 5 + 2\eta\right]x + \frac{1}{384}\left[[7560 + (615\pi^2 - 23\,200)\eta - 720\eta^2]\left(\frac{k'}{x}\right)^2\right.\right.$$

$$\left.\left. + [-5880 + (11\,072 - 123\pi^2)\eta - 960\eta^2]\left(\frac{k'}{x}\right) - 480 + 160\eta + 80\eta^2\right]x^2\right\}, \quad (28b)$$

where $x = [(GMn)/c^3]^{2/3}$ and $k' = (\frac{\Phi - 2\pi}{6\pi})$. The above equations generalize to eccentric orbits the gauge invariant expression connecting the orbital energy of a compact binary in circular orbit to its period at 3PN order [31,33]. Using 3PN accurate far-zone fluxes, which are not yet computed, and Eqs. (28a) and (28b), it should be possible to compute 3PN accurate expressions for dx/dt and dk'/dt in terms of x and k'. This would generalize the 3PN accurate gauge invariant expression for the rate of change of the orbital frequency of a compact binary in circular orbit, available in [4,33]. Equations (28a) and (28b) also indicate that it should be possible to characterize noncircular orbits in post-Newtonian relativity in an invariant manner. Therefore, this parametrization should be useful to analyze existing general relativistic simulations involving compact binaries in a post-Newtonian framework.

V. SUMMARY AND DISCUSSION

In this paper, we have presented the third post-Newtonian accurate Keplerian-type parametrization for the motion of two nonspinning compact objects moving in an eccentric orbit. The above 3PN accurate parametrization, which is structurally quite similar to the 2PN accurate generalized quasi-Keplerian representation, is given in both ADM-type and harmonic coordinates. The associated orbital elements and functions were explicitly computed in terms of the 3PN accurate conserved orbital energy, angular momentum, and the finite mass ratio. We explicitly showed that to even these high post-Newtonian orders there are gauge invariant quantities to characterize an eccentric orbit in post-Newtonian relativity. We also performed, for the first time, some clever consistency checks to validate the lengthy algebraic and trigonometric manipulations involved in the derivation of the parametrization.

There are quite a few possible applications for our parametrization and some of them are currently under investigation. The representation will be required to construct ready to use search templates for inspiraling eccentric compact binaries, whose orbital dynamics is fully 3.5PN accurate, extending the currently available 2.5PN accurate ones, presented in [5]. Recently, using a solution to 3PN accurate equations of motion for compact binaries moving in noncircular orbits, gauge dependent expressions for the associated conserved orbital energy and angular momentum were obtained which were used to analyze general relativistic simulations involving compact binaries [7,8]. Using the arguments that there are gauge invariant expressions to 3PN order in our parametrization and that the orbital elements of the representation, when radiation reaction is included, are continuously evolving variables, it should be possible to construct a better post-Newtonian accurate diagnostic tool to analyze the orbital configuration and motion of inspiralling compact binaries.

Finally, we note that, to obtain $\mathcal{O}(1/c^{11})$ corrections to the orbital averaged expressions for the far-zone fluxes associated with inspiralling eccentric binaries, 3PN corrections to the quadrapole approximation, our parametrization in harmonic coordinates will have to be heavily employed. The other crucial ingredients required for those computations, namely, the $\mathcal{O}(1/c^{11})$ corrections to the far-zone fluxes for compact binaries in general orbits, in harmonic coordinates, are expected to be available in the near future [4].

APPENDIX A: THE EQUATIONS GOVERNING THE 3PN ACCURATE RADIAL AND ANGULAR MOTION IN ADM-TYPE AND HARMONIC COORDINATES

In this appendix, we display the 3PN accurate expressions for $\dot{r}^2 = (1/s^4)(\frac{ds}{dt})^2$ and $\frac{d\phi}{ds} = \dot{\phi}/\frac{ds}{dt}$ that are essential to obtain the generalized quasi-Keplerian parametrization in ADM-type and harmonic coordinates in terms of $E, h, \eta,$ and s. As mentioned earlier, the expression for $\dot{r}^2 = (1/s^4)(\frac{ds}{dt})^2$ is required to obtain 3PN accurate turning points of the radial motion and, hence, required in the computations that determine the 3PN accurate radial parametrization and Kepler equation. We first exhibit the 3PN accurate expression for $\dot{r}^2$ in ADM-type coordinates:

$$\dot{r}^2 \equiv \frac{1}{s^4}\left(\frac{ds}{dt}\right)^2 = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + a_6 s^6 + a_7 s^7, \tag{A1a}$$

$$a_0 = 2E + \frac{3E^2}{c^2}(-1+3\eta) + \frac{E^3}{c^4}(4-19\eta+16\eta^2) + \frac{E^4}{4c^6}(-20+128\eta-211\eta^2+56\eta^3), \tag{A1b}$$

$$a_1 = 2 + \frac{2E}{c^2}(-6+7\eta) + \frac{6E^2}{c^4}(3-16\eta+7\eta^2) + \frac{2E^3}{c^6}(-12+93\eta-171\eta^2+35\eta^3), \tag{A1c}$$

$$a_2 = -h^2 + \frac{1}{c^2}[5(-2+\eta)+2h^2E(1-3\eta)] + \frac{1}{c^4}[E(37-122\eta+36\eta^2)+3E^2h^2(-1+5\eta-5\eta^2)]$$
$$\times + \frac{1}{2c^6}[E^2(-111+1034\eta-1268\eta^2+228\eta^3)+2E^3h^2(4-27\eta+50\eta^2-20\eta^3)], \tag{A1d}$$

$$a_3 = \frac{h^2}{c^2}(8-3\eta) + \frac{1}{2c^4}[(53-83\eta+20\eta^2)+2h^2E(-16+73\eta-19\eta^2)] + \frac{1}{24c^6}$$
$$\times\{E[-1896-(3\pi^2-14252)\eta-10848\eta^2+1776\eta^3]+12E^2h^2(48-348\eta+603\eta^2-96\eta^3)\}, \tag{A1e}$$

$$a_4 = \frac{3h^2}{c^4}(-11+11\eta-2\eta^2) + \frac{1}{48c^6}\{[-2400+(57\pi^2+9524)\eta-5292\eta^2+768\eta^3]$$
$$+12h^2E(264-1857\eta+1206\eta^2-120\eta^3)\}, \tag{A1f}$$

$$a_5 = -\frac{h^4}{4c^4}(4\eta+\eta^2) + \frac{1}{96c^6}\{h^2[9024+(-21908+3\pi^2)\eta+7872\eta^2-192\eta^3]+24Eh^4(12\eta-34\eta^2-27\eta^3)\},$$

$$a_6 = \frac{h^4}{4c^6}(71\eta+2\eta^2-32\eta^3), \quad a_7 = \frac{13h^6}{8c^6}\eta^3. \tag{A1g}$$

To obtain the post-Newtonian accurate Kepler equation and to parametrize the angular motion, we usually employ the PN accurate expressions for $\frac{ds}{dt}$ factorized by the PN accurate roots s_+ and s_-.

Instead of displaying the original expression for $\frac{d\phi}{ds} = \dot{\phi}/\frac{ds}{dt}$, we exhibit below the expression for $\frac{d\phi}{ds}$ factorized by the PN accurate roots s_+ and s_- which was actually employed in the computations:

$$\frac{d\phi}{ds} = -\frac{B_0 + B_1 s + B_2 s^2 + B_3 s^3 + B_4 s^4 + B_5 s^5}{\sqrt{(s_- - s)(s - s_+)}}, \tag{A2a}$$

where PN accurate functions B_i are given by

$$B_0 = 1 + \frac{1}{2h^2c^2}(6-\eta) + \frac{1}{c^4}\left[\frac{1}{8h^4}(176-80\eta+3\eta^2)+\frac{E}{2h^2}(15-12\eta+2\eta^2)\right] + \frac{1}{c^6}\left[\frac{1}{16h^6}[3376+(40\pi^2-3784)\right.$$
$$\left.\times\eta+350\eta^2-5\eta^3]+\frac{E}{48h^4}[6120+(57\pi^2-9592)\eta+1884\eta^2-72\eta^3]+\frac{E^2}{4h^2}(15-19\eta+32\eta^2-6\eta^3)\right], \tag{A2b}$$

$$B_1 = \frac{1}{2c^2}\eta + \frac{1}{c^4}\left[\frac{1}{4h^2}(17-5\eta-\eta^2)+\frac{E}{4}(2\eta+5\eta^2)\right] + \frac{1}{c^6}\left[\frac{1}{16h^4}[740+(20\pi^2-1114)\eta+6\eta^2+3\eta^3]\right.$$
$$\left.+\frac{E}{96h^2}[864+(-1708-3\pi^2)\eta+480\eta^2-12\eta^3]+\frac{E^2}{8}(5\eta^2-\eta^3)\right], \tag{A2c}$$

$$B_2 = \frac{1}{8c^4}(4\eta+17\eta^2) + \frac{1}{c^6}\left[\frac{1}{16h^2}[64+(10\pi^2-338)\eta+86\eta^2-17\eta^3]+\frac{E}{8}(-33\eta-28\eta^2+45\eta^3)\right], \tag{A2d}$$

$$B_3 = -\frac{9h^2}{8c^4}\eta^2 + \frac{1}{c^6}\left[\frac{1}{64}[(-100+3\pi^2)\eta-568\eta^2+504\eta^3]-4h^2E\eta^3\right], \tag{A2e}$$

$$B_4 = \frac{h^2}{16c^6}(20\eta+112\eta^2-121\eta^3), \tag{A2f}$$

$$B_5 = \frac{15h^4}{16c^6}\eta^3. \tag{A2g}$$

In harmonic coordinates, the explicit expressions for $\dot{r}^2$ and $\dot{\phi}$ are extracted from the 3PN accurate expressions for conserved orbital energy and angular momentum, in the center-of mass frame, given by Eqs. (22) and (23). To 3PN order, $\dot{r}^2$ in harmonic coordinates reads

$$\dot{r}^2 \equiv \frac{1}{s^4}\left(\frac{ds}{dt}\right)^2 = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + a_6 s^6 + a_7 s^7, \tag{A3a}$$

$$a_0 = 2E + \frac{3E^2}{c^2}(-1 + 3\eta) + \frac{E^3}{c^4}(4 - 19\eta + 16\eta^2) + \frac{E^4}{4c^6}(-20 + 128\eta - 211\eta^2 + 56\eta^3), \tag{A3b}$$

$$a_1 = 2 + \frac{2E}{c^2}(-6 + 7\eta) + \frac{6E^2}{c^4}(3 - 16\eta + 7\eta^2) + \frac{2E^3}{c^6}(-12 + 93\eta - 171\eta^2 + 35\eta^3), \tag{A3c}$$

$$a_2 = -h^2 + \frac{1}{c^2}[5(-2 + \eta) + 2h^2(1 - 3\eta)E] + \frac{1}{c^4}[E(36 - 127\eta + 36\eta^2) + 3h^2 E^2(-1 + 5\eta - 5\eta^2)]$$

$$+ \frac{1}{c^6}\left[\frac{E^2}{6}(-324 + 3476\eta - 4017\eta^2 + 684\eta^3) + h^2 E^3(4 - 27\eta + 50\eta^2 - 20\eta^3)\right], \tag{A3d}$$

$$a_3 = \frac{h^2}{c^2}(8 - 3\eta) + \frac{1}{c^4}\left[\frac{1}{2}(52 - 81\eta + 20\eta^2) + h^2 E(-16 + 61\eta - 19\eta^2)\right]$$

$$+ \frac{1}{c^6}\left[\frac{E}{840}[-63840 + (661412 + 4305\pi^2)\eta - 400\,260\eta^2 + 62\,160\eta^3]\right.$$

$$\left. + \frac{h^2 E^2}{2}(48 - 288\eta + 399\eta^2 - 96\eta^3)\right], \tag{A3e}$$

$$a_4 = \frac{h^2}{4c^4}(-132 + 75\eta - 24\eta^2) + \frac{1}{c^6}\left[\frac{1}{840}[-42\,000 + (241\,364 + 4305\pi^2)\eta - 93\,450\eta^2\right.$$

$$\left. + 13\,440\eta^3] + \frac{h^2 E}{4}(264 - 1887\eta + 552\eta^2 - 120\eta^3)\right], \tag{A3f}$$

$$a_5 = \frac{h^4}{4c^4}(15\eta - \eta^2) + \frac{1}{c^6}\left[\frac{h^2}{1120}[107\,520 + (-1435\pi^2 - 259\,344)\eta + 22\,960\eta^2 - 2240\eta^3]\right.$$

$$\left. + \frac{h^4 E}{4}(-71\eta + 241\eta^2 - 27\eta^3)\right], \tag{A3g}$$

$$a_6 = \frac{h^4}{8c^6}(-11\eta + 358\eta^2 - 64\eta^3), \tag{A3h}$$

$$a_7 = \frac{h^6}{8c^6}(23\eta - 73\eta^2 + 13\eta^3). \tag{A3i}$$

As in ADM-type gauge, the 3PN accurate factorized expression for $\frac{d\phi}{ds}$ reads

$$\frac{d\phi}{ds} = -\frac{B_0 + B_1 s + B_2 s^2 + B_3 s^3 + B_4 s^4 + B_5 s^5}{\sqrt{(s_- - s)(s - s_+)}}. \tag{A4a}$$

The explicit expressions for the B_i in harmonic coordinates are

$$B_0 = 1 + \frac{1}{2h^2c^2}(6 - \eta) + \frac{1}{c^4}\left[\frac{1}{8h^4}(172 - 148\eta + 3\eta^2) + \frac{E}{4h^2}(28 - 63\eta + 4\eta^2)\right] + \frac{1}{c^6}\left\{\frac{1}{1680h^6}[341880 + (8610\pi^2\right.$$

$$-640588)\eta + 74970\eta^2 - 525\eta^3] + \frac{E}{840h^4}[99960 + (4305\pi^2 - 345446)\eta + 76125\eta^2 - 1260\eta^3]$$

$$+\frac{E^2}{24h^2}(84 - 1055\eta + 567\eta^2 - 36\eta^3)\Big\}, \tag{A4b}$$

$$B_1 = \frac{1}{2c^2}\eta + \frac{1}{c^4}\left[\frac{1}{4h^2}(16 - 22\eta - \eta^2) + \frac{5E}{4}(\eta + \eta^2)\right] + \frac{1}{c^6}\left\{\frac{1}{1680h^4}[73920 + (-196604 + 4305\pi^2)\eta + 5460\eta^2\right.$$

$$+315\eta^3] + \frac{E}{3360h^2}[26880 + (-191048 + 4305\pi^2)\eta + 31080\eta^2 - 420\eta^3] + \frac{E^2}{8}(-9\eta + 28\eta^2 - \eta^3)\Big\}, \tag{A4c}$$

$$B_2 = \frac{1}{8c^4}(4 + 67\eta + 17\eta^2) + \frac{1}{c^6}\left\{\frac{1}{3360h^2}[18480 + (4305\pi^2 - 8444)\eta - 1050\eta^2 - 3570\eta^3]\right.$$

$$+\frac{E}{24}(134\eta + 411\eta^2 + 135\eta^3)\Big\}, \tag{A4d}$$

$$B_3 = \frac{h^2}{8c^4}(3\eta - 9\eta^2) + \frac{1}{c^6}\left\{\frac{1}{6720}[(-56296 - 12915\pi^2)\eta + 103320\eta^2 + 52920\eta^3] + \frac{h^2E}{4}(\eta + 4\eta^2 - 16\eta^3)\right\}, \tag{A4e}$$

$$B_4 = \frac{h^2}{48c^6}(581\eta - 45\eta^2 - 363\eta^3), \tag{A4f}$$

$$B_5 = \frac{5h^4}{16c^6}(\eta - 5\eta^2 + 5\eta^3). \tag{A4g}$$

We note that the expressions for $\dot{r}^2$ and $\frac{d\phi}{ds}$ in these two gauges have the same structure though the coefficients differ at 2PN and 3PN orders.

APPENDIX B. A SKETCH OF SOME COMPUTATIONAL DETAILS

This appendix details the computation required to obtain the 3PN accurate Kepler equation and the parametrization for the angular motion. As explained in Sec. III, the temporary form for the 3PN accurate $l(u)$ relation, given by Eq. (13), follows from Eqs. (9) and (12) by employing the auxiliary variable $\tilde{v}$. This computation is not straightforward as the direct evaluation of the integral for $(t - t_0)$, given by Eq. (13), gives a PN accurate expression in terms of E, h, η, s_-, s_+, and s. To obtain the temporary form for the 3PN accurate Kepler equation, we multiply the above result with the 3PN accurate expression for n, and use the trigonometric relations given below:

$$s = \frac{1}{a_r[1 - e_r\cos(u)]} = \frac{1 + e_r\cos\tilde{v}}{a(1 - e_r^2)}, \tag{B1a}$$

$$u = \arccos\left(\frac{s_- + s_+}{s_- - s_+} - 2\frac{s_-s_+}{(s_- - s_+)s}\right), \tag{B1b}$$

$$\tilde{v} = \arccos\left(\frac{2s}{s_- - s_+} - \frac{s_- + s_+}{s_- - s_+}\right). \tag{B1c}$$

These relations are also employed heavily to obtain the temporary parametrization for the angular motion, given by Eq. (16).

Let us turn our attention to the details of the computation, which gave the final parametrization for $\frac{2\pi}{\Phi}(\phi - \phi_0)$. The starting points of the above calculation are Eq. (16) and the introduction of PN accurate true anomaly $v = 2\arctan\{[(1 + e_\phi)/(1 - e_\phi)]^{1/2}\tan\frac{u}{2}\}$, where e_ϕ differs from e_r by yet to be determined PN corrections. It is easy to obtain the following 3PN accurate expression for $\tilde{v}$ in terms of v, which reads

$$\tilde{v} = v - \frac{y}{c^2}\sin v + \frac{y^2}{4c^4}(\sin 2v + 2\sin v) - \frac{y^3}{12c^6}(3\sin v$$
$$+ 3\sin 2v + \sin 3v), \tag{B2}$$

where y is expressible in terms of those PN corrections which relate e_ϕ and e_r. Using the above relation, we express $\frac{2\pi}{\Phi}(\phi - \phi_0)$, given by Eq. (16), in terms of v and demand that there are no $\sin v$ terms to 3PN order. This requirement, as explained in Sec. III, is motivated by the desire that we want for the first post-Newtonian order Keplerian-like parametrization for the angular part. This procedure uniquely defines PN corrections connecting e_ϕ to e_r and leads us to the final parametrization for the angular motion, given by Eq. (17).

Finally, we rewrite the temporary parametrization for the $l(u)$ relation, given by Eq. (13), in terms of v which leads us to the final expression for the 3PN accurate Kepler equation as given by Eq. (18).

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